Polynomials on Graphs with Applications

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Weighted Tutte Polynomial

Definition 1

For a Graph Γ we define the weighted Tutte polynomial as,

$$T_\Gamma(x_e, y_e, x, y) = \sum_{F \subseteq E(\Gamma)} \left(\prod_{e \in F} x_e \right) \Big(\prod_{e \in E(\Gamma) \setminus F} y_e \Big) (x-1)^{r(\Gamma)-r(F)} (y-1)^{n(F)}$$

Le $v(F)$ be the number of verticies of F, $e(F)$ be the number of edges of F, and $k(F)$ be the number of connected components of F. Define $r(F) := v(f) - k(f)$ to be the rank of the graph F and $n(F) := e(F) - r(F)$ to be the nullity of the graph F.

This is a generalization that agrees with the Tutte polynomial when we make the substitution $x_e = y_e = 1$. This definition also has the advantage that for any Ribbon Graph G we can consider its underlying Graph Γ and get by definition

$$R_G(x_e, y_e, x-1, y-1, 1) = T_\Gamma(x_e, y_e, x, y)$$

Signed Tutte polynomial

Definition 2

For a Graph Γ we define the Signed Tutte polynomial obtained by substitiuting

$$x_e = \begin{cases} 1 & \text{if } \operatorname{sign}(e) = + \\ \sqrt{\frac{x-1}{y-1}} & \text{if } \operatorname{sign}(e) = - \end{cases}$$

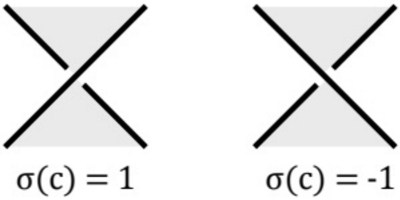
$$y_e = \begin{cases} 1 & \text{if } \operatorname{sign}(e) = + \\ \sqrt{\frac{y-1}{x-1}} & \text{if } \operatorname{sign}(e) = - \end{cases}$$

into $T_\Gamma(x_e, y_e, x, y)$ to get the Signed Tutte polynomial $\mathcal{T}_\Gamma(x, y)$.

This gives that $R_G(x-1, y-1, 1) = \mathcal{T}_\Gamma(x, y)$ where here $R_G(X, Y, Z)$ is the signed Bollobás-Riordan polynomial.

Signed Tait Graph

For a Checkerboard colorable virtual link diagram D we can assign a sign $\sigma(c)$ to each crossing c of D in the following way:



The sign on the crossings of D induces a sign on the edges Tait Graph. This gives us the Signed Tait graph of a checkerboard colorable virtual link diagram D.

Thistlethwaite's Theorem

Theorem 3 (Thistlethwaite)

Let $L \subset \mathbb{R}^2$ be a non-split link diagram and let Γ be a Signed Tait Graph of L then we have that,

$$\langle L\rangle(t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}) = t^{\frac{-2k+2v-e}{4}} \mathcal{T}_\Gamma(-t^{-1}, -t)$$

where $\langle L\rangle(A, B, d)$ is the Kauffman Bracket of L.

Corollary 4

Let $L \subset \mathbb{R}^2$ be a non-split link diagram and let Γ be a Tait Graph of L then we have that,

$$J_L(t) = (-1)^{w(L)} t^{\frac{-2k+2v-e+3w(L)}{4}} \mathcal{T}_\Gamma(-t^{-1}, -t)$$

where $J_L(t)$ is the Jones polynomial for the Link L and $w(L)$ is the writhe of L.

Example 1

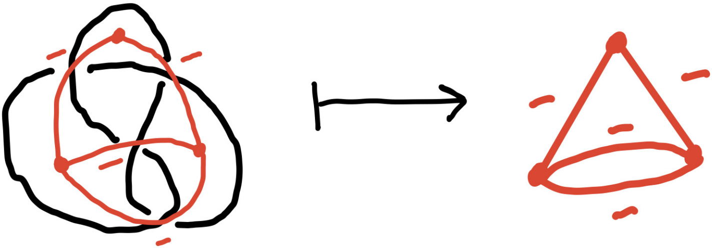
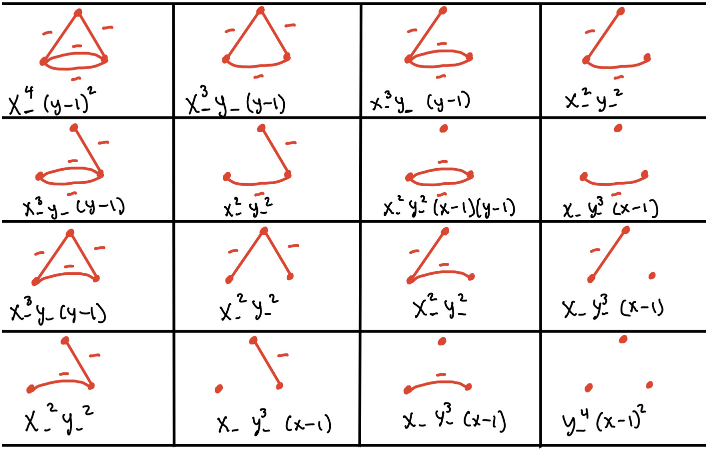


Figure: Signed Tait Graph of 4_1 Knot

Example 1


$x_-^4 (y-1)^2$
$x_-^3 y_- (y-1)$
$x_-^3 y_- (y-1)$
$x_-^2 y_-^2$
$x_-^3 y_- (y-1)$
$x_-^2 y_-^2$
$x_-^2 y_-^2 (x-1)(y-1)$
$x_- y_-^3 (x-1)$
$x_-^3 y_- (y-1)$
$x_-^2 y_-^2$
$x_-^2 y_-^2$
$x_- y_-^3 (x-1)$
$x_-^2 y_-^2$
$x_- y_-^3 (x-1)$
$x_- y_-^3 (x-1)$
$y_-^4 (x-1)^2$

Example 1

This Gives,

$$\mathcal{T}_\Gamma(x,y) = x_-^4(y-1)^2 + 4x_-^3y_-(y-1) + x_-^2y_-^2(5+(x-1)(y-1))$$

$$+4x_-y_-^3(x-1) + y_-^4(x-1)^2$$

Since $-2k+2v-e = 2-6+4 = 0$ and $w(K) = 0$,

$$J_{4_1}(t) = t^{\frac{-2k+2v-e}{4}}\mathcal{T}_\Gamma(-t^{-1},-t) = \mathcal{T}_\Gamma(-t^{-1},-t)$$

$$= t^{-2}(-t-1)^2 + 4t^{-1}(-t-1) + 5 + (-t^{-1}-1)(-t-1)$$

$$+4t(-t^{-1}-1) + t^2(-t^{-1}-1)^2$$

$$= t^2 - t + 1 - t^{-1} + t^{-2}$$

Contraction-Deletion on Graphs

Two of the important operations on Graphs are the contraction and deletion operations. The contraction with respect to an edge e is denoted G/e. The deletion with respect to an edge e is denoted $G - e$.

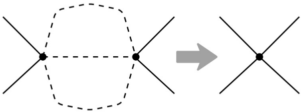


Figure: Contraction of edges

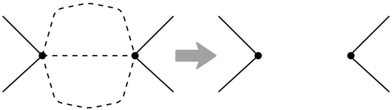


Figure: Deletion of edges

Alternate Definition For Weighted Tutte Polynomial

It is often useful to define polynomials on Graphs recursively by Contraction and Deletion Operations to help compute these polynomials.

Definition 5

We can Define the Weighted Tutte Polynomial, $T_G(x_e, y_e, x, y)$ to be the unique polynomial such that.

$$\begin{aligned}
T_\Gamma &= y_e T_{\Gamma - e} + x_e T_{\Gamma/e} && \text{If } e \text{ is not a bridge nor loop;}\\
T_\Gamma &= (y_e(x-1) + x_e) T_{\Gamma/e} && \text{If } e \text{ is a bridge;}\\
T_\Gamma &= (y_e + (y-1)x_e) T_{\Gamma - e} && \text{If } e \text{ is a loop;}\\
T_{\Gamma_1 \sqcup \Gamma_2} &= T_{\Gamma_1} \cdot T_{\Gamma_2} && \text{For disjoint union } \Gamma_1 \sqcup \Gamma_2;\\
T_\bullet &= 1
\end{aligned}$$

Thistlethwaite's Polynomial

Definition 6

In [TH], Thistlethwaite defined the Laurent polynomial $\tau[\Gamma]$ of a Graph Γ recursively by;

$$
\begin{array}{ll}
\tau[\Gamma] = A_e^{-1}\tau[\Gamma - e] + A_e\tau[\Gamma/e] & \text{If } e \text{ is not a bridge nor loop;} \\
\tau[\Gamma] = A_e^{-3}\tau[\Gamma/e] & \text{If } e \text{ is a bridge;} \\
\tau[\Gamma] = A_e^{3}\tau[\Gamma - e] & \text{If } e \text{ is a loop;} \\
\tau[\Gamma_1 \sqcup \Gamma_2] = d\tau[\Gamma_1]\tau[\Gamma_2] & \text{For disjoint union } \Gamma_1 \sqcup \Gamma_2; \\
\tau[\cdot] = 1 &
\end{array}
$$

Where $d = -A^2 - A^{-2}$ and $A_e = \begin{cases} A & \text{if } \mathrm{sign}(e) = + \\ A^{-1} & \text{if } \mathrm{sign}(e) = - \end{cases}$

Connection to Bollobás-Riordan Polynomial

From this definition of Thistlethwaite's polynomial it is not obvious that it is a specialization of the Bollobás-Riordan Polynomial. It is a straightforward verification that for a connected graph Γ we have,

$$\tau[\Gamma] = A^{2k-2v+e} \mathcal{T}_{\Gamma}(-A^4, -A^{-4})$$

by seeing that both polynomials are defined the same recursively. This means that,

$$\tau[\Gamma] = A^{2k-2v+e} R_G(-A^4 - 1, -A^{-4} - 1, 1)$$

One more Polynomial from a Graph

Definition 7

Let D be a checkerboard colorable Virtual link diagram and Γ be the associated Signed Tait Graph then we can define;

$$\nu_{D,\Gamma}(t) = \left((-A)^{-3w(D)} \tau[\Gamma] \right)_{A^{-2}=t^{1/2}}$$

It is clear from that this polynomial must be a specialization of the Bollobás-Ribbon polynomial.

A Polynomial Invariant

Theorem 8 (Boninger 23')

Let Σ be a closed orientable surface. Let $D \subset \Sigma$ be a checkerboard colorable, non-split link diagram, and $L \subset \Sigma \times I$ the associated link. Let Γ and Γ' be the signed Tait graphs associated to the two checkerboard colorings of D. Then,

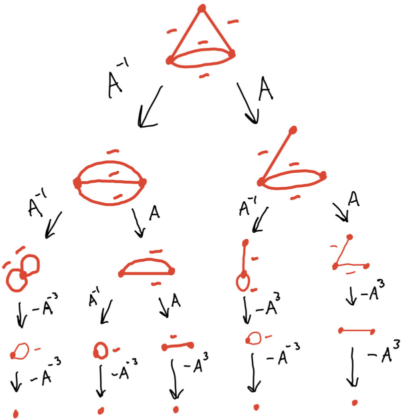
$$\{\nu_{D,\Gamma}(t), \nu_{D,\Gamma'}(t)\}$$

Is an isotopy invariant of L.

Note that from the definition of $\nu_{D,\Gamma}$ we can see that Thistlethwaite's theorem implies that for classical links we have $\nu_{D,\Gamma} = \nu_{D,\Gamma'} = J_D(t)$

Example 2

Let us compute $\tau[\Gamma]$ for the tait graph Γ of 4_1 we used earlier.



Example 2

We can see that from this Contraction-Deletion binary Tree that,

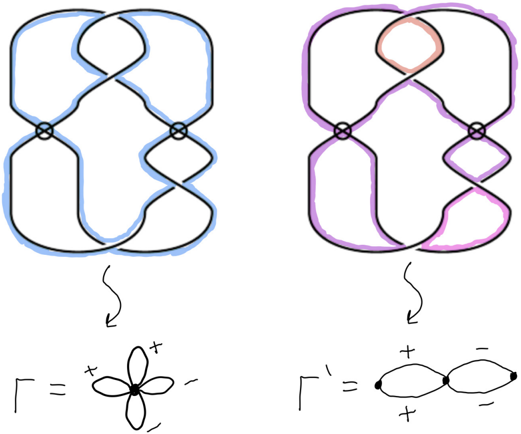
$$\tau[\Gamma] = A^{-8} - A^{-4} - A^{4} + 1 + A^{8}$$

Since we have $w(K) = 0$,

$$\nu_{4_1,\Gamma}(t) = t^2 - t + 1 - t^{-1} + t^{-2} = J_{4_1}(t)$$

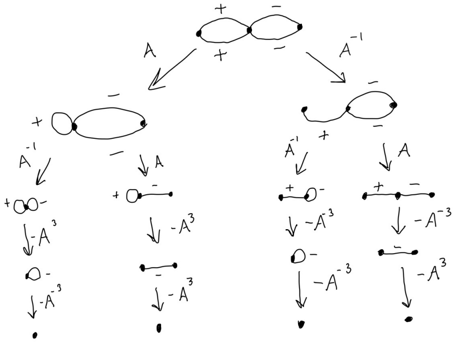
Example 3

Now let us Compute an example of a Virtual knot.



Example 3

It is easy to see that $\tau[\Gamma] = 1$, so let us compute $\tau[\Gamma']$



This Gives $\tau[\Gamma'] = A^{-8} + 2 + A^{8}$

Example 3

Now we have that $w(D) = 0$ which means,

$$\nu_{D,\Gamma}(t) = 1$$

and

$$\nu_{D,\Gamma'}(t) = t^{-2} + 2 + t^2$$

hence,

$$S = \{1, t^{-2} + 2 + t^2\}$$

Is an isotopy invariant of K. Note that since D is not a planar $J_D(t)$ need not be an element of S. This example illustrates that fact because we can compute

$$J_D(t) = t^2 - t + 1 - t^{-1} + t^{-2} \notin S$$

References

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KA L.H. Kauffman, A Tutte polynomial for signed graphs, Discrete Appl. Math. 25 (1989) 105–127.